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Fermionic zero modes of supergravity cosmic strings

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Abstract: Recent developments in string theory suggest that cosmic strings could be formed at the end of brane inflation. Supergravity provides a realistic model to study the properties of strings arising in brane inflation. Whilst the properties of cosmic strings in flat space-time have been extensively studied there are significant complications in the presence of gravity. We study the effects of gravitation on cosmic strings arising in supergravity. Fermion zero modes are a common feature of cosmic strings, and generically occur in supersymmetric models. The corresponding massless currents can give rise to stable string loops (vortons). The vorton density in our universe is strongly constrained, allowing many theories with cosmic strings to be ruled out. We investigate the existence of fermion zero modes on cosmic strings in supergravity theories. A general index theorem for the number of zero modes is derived. We show that by including the gravitino, some (but not all) zero modes disappear. This weakens the constraints on cosmic string models. In particular, winding number one cosmic D-strings in models of brane inflation are not subject to vorton constraints. We also discuss the effects of supersymmetry breaking on cosmic D-strings.

Keywords: String theory and cosmic strings, Supergravity Models, Solitons Monopoles and Instantons.

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1. Introduction

There has been a resurgence of interest in cosmic strings arising from recent results in fundamental superstring theory (for reviews see [1, 2]). Indeed, in string theory a generic feature of D-brane anti-brane annihilation is the production of lower dimensional branes, with $D3$ and $D1$ branes, or D-strings, favoured [3]. Consequently, in models of brane inflation where a period of inflation is caused by the attraction, and subsequent annihilation, of a D-brane and anti-brane, D-strings form naturally [4]. Depending on the details of the theory, fundamental strings, or F-strings can also arise [5] and, in certain classes of models, axionic local strings [6]. The D-strings arising in such models have many features in common with cosmic strings formed in a cosmological phase transition (for reviews see [7, 8]). However, there are added complications arising from supergravity which need to be addressed.

Cosmic strings in supersymmetric theories have been investigated and shown to give rise to two sorts of strings, called D-term and F-term strings [9], where the D and F refer to the type of potential required to break the symmetry. For the D-term case, this has been extended to supergravity [10], and an analysis of the BPS configurations led to the conjecture that the D-term strings of supergravity were in fact D-strings [11]. Consequently, an analysis of strings in supergravity theories should provide insight into those formed at the end of brane inflation.

Strings arising in supersymmetric theories have fermion zero modes in the string core [9]. This is an inevitable result of the couplings and particle content required by the supersymmetry algebra. In certain cases the zero modes survive supersymmetry breaking [12]. The presence of fermion zero modes changes the cosmology of cosmic strings drastically since they result in the string carrying a current[1]. Ordinary cosmic strings are either long strings or closed loops. The latter usually decay via the emission of gravitational radiation, resulting in a scaling solution of the string network. However, when the string carries a current the loop can be stabilised by the current carriers [14]. These stable loops, or vortons, put constraints on the underlying theory [15, 16]. The presence of zero modes on cosmic strings has been well studied in the case of global supersymmetry [9]. An important question to ask is whether these results go over to the case of supergravity. An analysis of supersymmetry transformations in ref. [17] suggests that conductivity is reduced for supergravity cosmic strings. In this paper we will address the question in detail.

In general supergravity contains both D- and F- terms, and cosmic strings can arise from both classes of theory, though only the former have been studied to date. Here we investigate both D- and F-term theories. This is particularly relevant since general brane inflation models contain both types of terms. Of course, only strings arising from the D-term would be BPS states. The paper is arranged as follows. In section 2 we consider the gravitational background created by cosmic strings. Here we solve for the most general metric appropriate to cylindrical symmetry, which will be used later when calculating string zero modes. In section 3 we prove a general index theorem for the Dirac operator in the cosmic string background including gravitation. We consider the most general form of mass matrices that can arise in such theories. Supersymmetric examples are reviewed in section 4, where we apply the index theorem to D- and F-term strings. In section 5, we

[1]Such strings are widely referred to as 'superconducting', when in fact they are actually just perfect conductors. To be superconducting they would not only need fermion zero modes, but the fermions would also have to form into Cooper pair bound states [13]. For cosmology the distinction between super- and perfect conductivity is not very important, since most of the cosmological properties of conducting strings follow from the fact that the currents are conserved (which is true for both types of current).

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consider the supergravity case. Here we map the gravitino equations onto Dirac equations amenable to a treatment as in section 3. Supergravity examples are considered in section 6. We find that the presence of the gravitino generically reduces the number of zero modes on supergravity cosmic strings. In particular, we find that for BPS D-strings the number of chiral zero modes are reduced from $2n$ in the global case to $2(n-1)$ in supergravity. Thus winding number 1 D-strings evade the stringent vorton constraints found for chiral theories [16]. In section 7, we extend our results on zero modes to the issue of massless currents on the cosmic string world–sheet. We conclude in section 8.

2. Gravitating cosmic strings

We consider a cosmic string configuration created by the spontaneous breaking of gauge symmetries. The simplest example of this would be a $U(1) \to I$ breaking, although any symmetry breaking whose vacuum manifold is not simply connected can produce cosmic string solutions. In the cosmic string background, the scalar fields have a profile

$$\phi^i(r,\theta) = e^{i\theta T_s}\phi^i(r) \tag{2.1}$$

where the string generator T_s is some linear combination of generators, T^u, from the broken gauge group. In this paper we will define it to include the winding number of the string. The covariant derivative of the scalar fields is

$$D_\mu\phi^i = (\partial_\mu - iT^u A^u_\mu)\phi^i\,. \tag{2.2}$$

This must vanish at infinity to ensure a finite energy solution. This is achieved by having

$$T^u A^u_\theta|_{r\to\infty} = T_s\,. \tag{2.3}$$

The simplest string gauge fields will only involve one generator, although more generally solutions can have several gauge fields, each with it own length scale. When there is only one generator, $T_s = nQ$ where Q is the electric charge and n is the string's winding number.

The gravitational effects of a cosmic string lead to a deficit angle in the far away metric of spacetime. In the following we will consider the metric

$$ds^2 = e^{2B}(-dt^2 + dz^2 + d\rho^2 + \alpha^2 d\theta^2) \tag{2.4}$$

for a cosmic string configuration. This is the most general cylindrically symmetric metric, as discussed by Thorne [18]. Notice that B is an overall conformal factor and α is related to the deficit angle of the cosmic strings. The Einstein equation involves the energy momentum tensor and reduces to

$$\begin{aligned} \alpha'' &= \kappa\alpha e^{2B}(T^t_t + T^\rho_\rho) \\ 2B'' &= -(B')^2 + \kappa T^\theta_\theta \\ (\alpha B')' &= \frac{\kappa}{2}\alpha e^{2B}(T^\rho_\rho + T^\theta_\theta)\,. \end{aligned} \tag{2.5}$$

These equations can be solved numerically by imposing that the metric is regular at the origin and satisfies $B'(0) = 0$, $\alpha(0) = 0$ and $\alpha'(0) = 1$. We also impose $B(\infty) = 0$.

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In order to determine of the number of fermion zero modes we will only require the form of the string solution at large and small r. Hence an approximate solution will be sufficient for our analysis. We will look for solutions using the top hat approximation whereby the scalars are assumed to be constant inside and outside the cosmic string with a jump at the string radius. Let us consider the solution inside. First of all, notice that the energy momentum tensor is almost constant with $T^\rho_\rho \approx T^\theta_\theta$. Let us define $m^2 = \kappa(T^t_t + T^\rho_\rho)$, then the metric inside reads

$$\alpha = \frac{\sin(m\rho)}{m} \tag{2.6}$$

while

$$B = -\frac{\kappa T^\rho_\rho}{m^2}\left(\ln\sin m\rho - m\int^\rho \frac{d\rho'}{\sin m\rho'}\right) + B_*\,. \tag{2.7}$$

A good approximation for small ρ is given by

$$B \approx \frac{\kappa}{4}T^\rho_\rho \rho^2 + B_* \tag{2.8}$$

satisfying the boundary conditions at the origin. Let us now consider the outside solution. In this region, the energy momentum tensor is approximately zero and therefore B is zero. Now we also find that

$$\alpha = C_0 + C_1\rho\,. \tag{2.9}$$

When $C_1 \neq 0$, the solution is a cosmic string solution with a deficit angle $\Delta = 2\pi(1 - C_1)$. Indeed, after defining $\tilde\rho = (\rho + C_0/C_1)$ the metric becomes

$$ds^2 = -dt^2 + dz^2 + d\tilde\rho^2 + C_1^2\tilde\rho^2 d\theta^2 \tag{2.10}$$

near infinity.

Using the change of variables $dr = e^B d\rho$ we obtain

$$ds^2 = e^{2B}(-dt^2 + dz^2) + dr^2 + C^2 d\theta^2 \tag{2.11}$$

which is the metric we will use for the rest of this paper. Near the origin $C \approx r$ and far from the string $C = C_1 r + C_0 + O(1/r)$.

In supersymmetric theories, a cosmic string breaks all supersymmetries in its core in general. BPS objects are an exception to this rule, as they leave 1/2 of the original supersymmetry unbroken. D-strings, in which we will be interested in this paper, are an example of this. These strings have vanishing T^ρ_ρ, and the conformal factor B is identically zero. As we will discuss in more detail in subsection 5.1, the string will only be BPS if the superpotential vanishes for the string solution. The only non-zero potential energy for the string comes from the D-term corresponding to T_s.

Let us characterise the BPS cosmic strings. We are considering a U(1) symmetry breaking, so we take $T_s\phi^i = nQ_i\phi^i$ and $A_\mu = \delta^\theta_\mu n a(r)$. The bosonic fields satisfy first order equations

$$\partial_r\phi^i = \mp n\frac{1-a}{C}Q_i\phi^i \tag{2.12}$$

and

$$\mp n\frac{\partial_r a}{C} = D = \xi - \sum_i Q_i K_i \phi^i \tag{2.13}$$

where ξ is the Fayet-Iliopoulos term which triggers the breaking of the U(1) gauge symmetry and K is the Kahler potential. The Einstein equations reduce to $B' = 0$ and

$$C' = 1 \pm A^B_\theta \tag{2.14}$$

where

$$A^B_\mu = \frac{i}{2}(\bar{K}_{\bar{j}} D_\mu \bar{\phi}^{\bar{j}} - K_j D_\mu \phi^j) + \xi A_\mu \,. \tag{2.15}$$

The simplest BPS configuration will just have one ϕ, with unit charge. Notice that BPS cosmic strings are solutions of first order differential equations. These equations are consequences of the Killing spinor equations when requiring the existence of 1/2 supersymmetry.

In order to make sure that the deficit angle of the string is positive ($C_1 < 1$), we must choose $\pm = -\text{sign}(n)$, which gives

$$C_1 = 1 - |n|\xi \,. \tag{2.16}$$

Notice that the deficit angle of the string is directly proportional to the winding number. Restricting ourselves to positive tension strings implies that the winding n cannot be arbitrarily large.

We will also consider the non-BPS configurations where the tension is higher than the BPS tension, this implies that C_1 is less than the BPS case

$$C_1 \leq 1 - |n|\xi \,. \tag{2.17}$$

We will see that this change in the deficit angle can alter the number of fermion zero modes on a string.

3. Zero modes and the index theorem

3.1 The Dirac equation

When fermions live in a cosmic string background, they are subject to both the electromagnetic interaction and the gravitational interaction. We consider a family of n_f Weyl fermions with a Lagrangian

$$\mathcal{L} = g_{ij} \bar{\chi}^i_{\dot{\alpha}} i \bar{\sigma}^{\dot{\alpha}\alpha}_\mu D^\mu \chi^j_\alpha + \frac{1}{2} \bar{\chi}^i_{\dot{\alpha}} M_{ij} \bar{\chi}^{j\dot{\alpha}} + (\text{c.c.}) \tag{3.1}$$

where $g_{ij}(\phi, \bar{\phi})$ is a sigma-model metric depending on the cosmic string background. The covariant derivative involves the gauge fields A^a_μ and the spin connection term w_μ

$$D_\mu \chi^i = (\partial_\mu + w_\mu)\,\chi^i - i A^u_\mu (T^u)^i_j \chi^j + \Gamma^i_{jk} D_\mu \phi^j \chi^k + J^i_{\mu j} \chi^j \tag{3.2}$$

$w_\mu = -w_{\mu ab}\sigma^{ab}/2$, where a, b are flat indices and $\sigma^{ab} = (\sigma^a \bar{\sigma}^b - \sigma^a \bar{\sigma}^b)/4$. In particular we find that $\sigma^{12} = -i\sigma^3/2$.

For a cosmic string, $w_{\mu ab}$ has only three non-zero entries

$$w_{t\hat{r}\hat{t}} = w_{z\hat{r}\hat{z}} = -B' , \qquad w_{\theta\hat{\theta}\hat{r}} = C' \tag{3.3}$$

where hatted quantities are flat indices and we identify $\hat{t} \equiv 0$, $\hat{r} \equiv 1$, $\hat{\theta} \equiv 2$ and $\hat{z} \equiv 3$.

We have also included the Levi-Civita connection Γ^i_{jk} to take into account the reparametrisation of the sigma-model scalar fields. Any other contributions are included in J_μ. Given the form of the string solution, we will assume J_z and J_t are zero. For the supergravity D-strings discussed in section 2 we find that

$$J_\mu = \frac{1}{4}(K_j D_\mu \phi^j - \bar{K}_{\bar{j}} D_\mu \bar{\phi}^{\bar{j}}) \tag{3.4}$$

and only J_θ is non-vanishing.

Notice that the Weyl fermions live in the spin-bundle of the sigma-model manifold defined by the scalar field, hence we have as many fermions as scalars in the curved case. We can have more fermions than scalar fields by considering fermions having a flat sigma-model metric. This happens in the supersymmetric setting with the Kahler manifold defining the set of scalar fields. The gauginos are not associated to the scalar fields and their sigma-model metric is flat.

In the following we will diagonalise the sigma-model metric using the vielbein e^a_i such that

$$g_{ij} = \delta_{ab} e^a_i e^b_j \tag{3.5}$$

and redefine the fermions

$$\chi^i = e^i_a \chi^a . \tag{3.6}$$

We also choose the χ^a to diagonalise the string generator, so that $T_s \chi^a = q_a \chi^a$.

The Lagrangian can then be written as

$$\mathcal{L} = \bar{\chi}^a i \bar{\sigma}^\mu D_\mu \chi^a + \frac{1}{2} \bar{\chi}^a M_{ab} \bar{\chi}^b + (\text{c.c.}) \tag{3.7}$$

where summation over the a, b indices is understood and

$$M_{ij} = M_{ab} e^a_i e^b_j . \tag{3.8}$$

The covariant derivative becomes

$$D_\mu \chi^a \to D_\mu \chi^a + e^a_j \left(\frac{\partial}{\partial \phi^k} \right) e^j_c D_\mu \phi^k \chi^c + e^a_j \left(\frac{\partial}{\partial \bar{\phi}^{\bar{k}}} \right) e^j_c D_\mu \bar{\phi}^{\bar{k}} \chi^c . \tag{3.9}$$

For a wide range of models this cancels the term arising from the Levi-Civita connection Γ^i_{jk} of the scalar manifold, although this is not always the case. An example where such a cancellation does occur has been given in ref. [6] for a Kahler metric $g_{ij} = \partial_i \partial_{\bar{j}} K$, where $K = -\ln(S + \bar{S})$ is the Kahler potential of the dilaton. In this case the two contributions from the non-flatness of the sigma model metric cancel in the cosmic string background. More generally, if any part of the connection is not cancelled, we absorb it into J_μ.

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The zero modes can be separated into σ^3 eigenstates. The upper component spinors will be denoted by χ^a_+ (for positive chirality), and the lower ones by χ^a_- (for negative chirality). The Dirac equations now read

$$\left(\partial_r + J_r + \frac{i}{C}\left[\partial_\theta - \frac{i}{2}C' - iT^u A^u_\theta + J_\theta\right]\right)\chi^a_+ - iM_{ab}(\chi^b_+)^* = 0 \tag{3.10}$$

and similarly

$$\left(\partial_r + J_r - \frac{i}{C}\left[\partial_\theta + \frac{i}{2}C' - iT^u A^u_\theta + J_\theta\right]\right)\chi^a_- + iM_{ab}(\chi^b_-)^* = 0\,. \tag{3.11}$$

Notice that the spin connection parts along the z and t directions cancel. We will look for normalisable solutions of these equations, and use the natural choice of norm

$$||\chi||^2 = \sum_a \int drd\theta C e^{2B}|\chi^a|^2\,. \tag{3.12}$$

As the B factor is constant at the origin and at infinity, we see that finiteness of the norm is equivalent to L^2 normalisability of the $\chi^a_\pm$.

Gauge invariance implies that the mass matrix must be factorisable

$$M_{ab} = e^{i(q_a+q_b)\theta}\mathcal{M}_{ab}(r)\,. \tag{3.13}$$

We can assume that both M_{ab} and $J^a_{\mu b}$ are not simultaneously block diagonal. If they are, we simply separate the fermions into several independent systems, and analyse each one individually.

As the mass matrix is obtained using algebraic combinations of the bosonic fields, it is a single-valued matrix. In general this implies that the fermion charges q_a must all be either integers or half integers, although there are some exceptions which require a different treatment. One exception is if the mass matrix is block off-diagonal. By this we mean that the only non-zero entries are in the $n_1 \times n_2$ upper-right and the $n_2 \times n_1$ lower left corners of the matrix (with $n_1 + n_2 = n_f$). This will be the case if system has only Dirac mass terms. It is also relevant for D-strings. Another exception to the above rule is if $M_{ab} \equiv 0$. In both these cases the charges are not automatically integers or half integers. We will deal with them in later subsections.

3.2 Generic mass matrix

The index theorem giving the number of normalisable zero modes is obtained by analysing the normalisability of the modes at the origin and at infinity. Close to the origin the metric is simply the flat metric in polar coordinates. One can decompose the fermions using the ansatz

$$\chi^a_\pm = \frac{1}{\sqrt{C}}e^{iq_a\theta}\left\{U^a_\pm(r)e^{\pm il\theta} \mp V^{a*}_\pm(r)e^{\mp il\theta}\right\}. \tag{3.14}$$

We require that the fermions are single valued when $\theta \to \theta + 2\pi$ around the origin. This implies that

$$q_a + l, q_a - l \in \mathbb{Z} + \frac{1}{2}\,. \tag{3.15}$$

These two conditions are satisfied provided l is an integer when the q_a are half-integer, or the converse. We will now use the approximate large and small r behaviour of the fermion field equations to determine the number of zero modes.

Close to the origin $A_\theta^u T^u$ vanishes, so to leading order

$$\partial_r U_\pm^a - \frac{1}{r}\left[l \pm q_a\right] U_\pm^a + i\mathcal{M}_{ab} V_\pm^b = 0 \tag{3.16}$$

and

$$\partial_r V_\pm^a + \frac{1}{r}\left[l \mp q_a\right] V_\pm^a + i\mathcal{M}_{ab} U_\pm^b = 0\,. \tag{3.17}$$

We have also assumed that J_μ is subdominant near the origin, as it is frequently the case in supergravity. If it is not subdominant, the results we are deriving will not be valid (although the approach we are using can be generalised).

Now since $\mathcal{M}_{ab} = O(1)$, this implies that the solutions behave as

$$U_\pm^a \sim r^{\pm q_a + l}\,, \qquad V_\pm^a \sim r^{\pm q_a - l} \tag{3.18}$$

where we have again assumed that J_μ can be neglected.

Normalisability of the above $\chi_\pm^a$ solutions at small r demands that either $l \geq \mp q_a + 1/2$ or $l \leq \pm q_a - 1/2$ for a given l. The total number of normalisable, small r solutions for a given l is then

$$N_0^\pm(l) = \sum_{i=1}^{n_f} \left\{ I\left(l \leq \pm q_a - \frac{1}{2}\right) + I\left(l \geq \mp q_a + \frac{1}{2}\right)\right\} \tag{3.19}$$

where I is the characteristic function of each interval (so $I(\text{true}) = 1$ and $I(\text{false}) = 0$).

At infinity the behaviour of the fermion fields is mainly determined by the mass matrix. Defining $\tilde{U}, \tilde{V}$ to be linear combinations of U and V which diagonalise $\mathcal{M}_{ab}$ at $r = \infty$, the asymptotic Dirac equation reads

$$\partial_r \tilde{U}_\pm^a - \frac{l}{C_1 r}\tilde{U}_\pm^a + i\lambda_a \tilde{V}_\pm^a = 0 \tag{3.20}$$

$$\partial_r \tilde{V}_\pm^a + \frac{l}{C_1 r}\tilde{V}_\pm^a + i\lambda_a \tilde{U}_\pm^a = 0 \tag{3.21}$$

where λ_a are the eigenvalues of the mass matrix. If all λ_a are non-zero, the $2n_f$ solutions will have exponential behaviour, and exactly half of them vanish at infinity, and so be normalisable.

On the other hand if some $\lambda_a = 0$ then the leading order behaviour of those solutions is instead

$$\tilde{U}_\pm^a \sim r^{l/C_1}\,, \qquad \tilde{V}_\pm^a \sim r^{-l/C_1}\,. \tag{3.22}$$

One of these will be normalisable if $|l| > C_1/2$. The total number of normalisable, large r solutions for a given l is therefore

$$N_\infty^\pm(l) = n_f - n_z I\left(|l| \leq \frac{C_1}{2}\right) \tag{3.23}$$

where n_z is the number of massless fermions at infinity, or equivalently the number zero eigenvalues λ_a. It will be useful to express eq. (3.23) in a similar form to eq. (3.19). If we define

$$\tilde{q}_\pm = \begin{cases} \mp\frac{1}{2} & \text{if } q_a \in \mathbb{Z}+1/2 \\ 0 & \text{if } q_a \in \mathbb{Z} \end{cases} \tag{3.24}$$

we can write

$$N_\infty^\pm(l) = n_f - n_z + n_z\left\{I\left(l \leq \pm\tilde{q} - \frac{1}{2}\right) + I\left(l \geq \mp\tilde{q} + \frac{1}{2}\right)\right\}. \tag{3.25}$$

We have explicitly taken into account the fact that $0 < C_1 < 1$ for a cosmic string with a non-vanishing deficit angle.

The set of normalisable zero modes is characterised by conditions at the origin and infinity. When the mass matrix is generic, the total number of zero mode solutions for a given l forms a vector space of dimension $2n_f$. Solutions which are normalisable at the origin form an $N_0^\pm(l)$-dimensional subspace of this. The solutions which are normalisable at infinity form an $N_\infty^\pm(l)$-dimensional subspace. Hence the number of independent solutions which are normalisable everywhere is equal to the dimension of the intersection of these two subspaces. This implies there are

$$N^\pm(l) = \left[N_0^\pm(l) + N_\infty^\pm(l) - 2n_f\right]_+ \tag{3.26}$$

solutions, where $[x]_+ = x$ if $x \geq 0$ and zero otherwise. Of course the above argument only gives a lower bound for the number of zero modes. It is possible there will be additional solutions, although this will generally require fine tuning of the string background. Extra solutions would also occur if the mass matrix M_{ab} has some degeneracy. For example if it is block diagonal. However we have already included this possibility in the analysis.

To simplify the above expression (3.26) it is convenient to separate the winding numbers q_a into n_+ positive values q_a^+ and n_- negative values q_a^-. In what follows we will concentrate on the χ_a^+ modes, although a similar analysis applies for χ_a^-. To take into account the effect of any massless fermions that are present, we define $\hat{q}_a^-$ to be the union of q_a^- and n_z copies of $\tilde{q}_+$. Note that $\tilde{q}_\pm$ has been defined so that, like the q_a, it will either be an integer or a half integer. Then

$$\begin{aligned} &N_0^+(l) + N_\infty^+(l) - 2n_f = \\ &\quad \sum_{a=1}^{n_+} I\left(-q_a^+ + \frac{1}{2} \leq l \leq q_a^+ - \frac{1}{2}\right) - \sum_{a=1}^{n_- + n_z} I\left(\hat{q}_a^- - \frac{1}{2} < l < -\hat{q}_a^- + \frac{1}{2}\right). \end{aligned} \tag{3.27}$$

Any zero q_a do not contribute to the sum and therefore do not lead to zero modes. We will place the charges in decreasing order of magnitude, so that $q_1^+ \geq \dots q_{n_+}^+$ and $\hat{q}_1^- \leq \dots \hat{q}_{n_-+n_z}^-$. This allows the above expression (3.27) to be rewritten as

$$\sum_{a=1}^{n_*}\left\{I\left(-q_a^+ + \frac{1}{2} \leq l \leq \hat{q}_a^- - \frac{1}{2}\right) + I\left(-\hat{q}_a^- + \frac{1}{2} \leq l \leq q_a^+ - \frac{1}{2}\right)\right.$$

$$-I\left(\hat{q}_a^- - \frac{1}{2} < l < -q_a^+ + \frac{1}{2}\right) - I\left(q_a^+ - \frac{1}{2} < l < -\hat{q}_a^- + \frac{1}{2}\right)\Bigg\}$$

$$+\sum_{a=n_*+1}^{n_+} I\left(-q_a^+ + \frac{1}{2} \le l \le q_a^+ - \frac{1}{2}\right) - \sum_{a=n_*+1}^{n_-+n_z} I\left(\hat{q}_a^- - \frac{1}{2} < l < -\hat{q}_a^- + \frac{1}{2}\right) \tag{3.28}$$

with $n_* = \min(n_+, n_- + n_z)$.

Taking into account the double counting due to the $l \to -l$ symmetry and the fact that only one of the solutions is non-vanishing for $l = 0$, we find that the number of real solutions is given by

$$N^\pm = \sum_l N^\pm(l) \tag{3.29}$$

with $N^\pm(l)$ given by the above expression (3.26). It can be seen that for a given l, the only non-zero terms in the sums (3.28) are either all positive or all negative. If they are all negative, the expression for $N^+(l)$ (3.26) evaluates to zero. Hence we can ignore all negative terms in the sum (3.28) when we evaluate $N^+(l)$. Our expression for the total number of zero modes reduces to a sum of only positive terms. We can therefore change the order of the l and a summations. From this we obtain

$$N^+ = 2\sum_{a=1}^{n_*}\left[q_a^+ + \hat{q}_a^-\right]_+ + 2\sum_{a=n_*+1}^{n_+} q_a^+ \tag{3.30}$$

where q_a^+ are all the positive q_a, and $\hat{q}_a^-$ are all the negative q_a and n_z copies of $\tilde{q}_+$. For the summations we have taken $\sum_{a=a_1}^{a_2} \ldots = 0$ if $a_1 > a_2$. The above expression is the generalisation of the index theorem to self-gravitating cosmic strings. It coincides with previous index theorems [19, 20] which apply when there are no massless fermions.

Using the fact that $|\tilde{q}_+| \le |q_a^-|$, we can rewrite the above sum as

$$N^+ = 2\sum_{a=1}^{\min(n_-,n_+)}\left[q_a^+ + q_a^-\right]_+ + 2\sum_{a=n_-+1}^{\min(n_+,n_-+n_z)}\left[q_a^+ + \tilde{q}_+\right]_+ + 2\sum_{a=n_-+n_z+1}^{n_+} q_a^+ \,. \tag{3.31}$$

A similar analysis applies to the other chirality. This time the roles of the q_a^+ and q_a^- are interchanged. If there are massless fermions present, we define $\hat{q}_a^+$ to be q_a^+ with n_z copies of $\tilde{q}_-$ added. Taking $n'_* = \min(n_-, n_+ + n_z)$, we obtain

$$N^- = 2\sum_{a=1}^{n'_*}\left[-\hat{q}_a^+ - q_a^-\right]_+ + 2\sum_{a=n'_*+1}^{n_-}(-q_a^-)\,. \tag{3.32}$$

From the above expressions, we find that if $n_+ = n_-$, the presence of massless fermions in the vacuum does not affect $N^\pm$.

3.3 Block off-diagonal or Dirac-like mass matrices

Now let us extend our results to Dirac-style mass matrices. The only non-zero elements of M_{ab} are in the $n_1 \times n_2$ upper-right corner and the $n_2 \times n_1$ lower-left corner of the matrix.

The total number of fermions is $n_f = n_1 + n_2$. The winding operator can be decomposed as

$$T_s = T_s^0 + T_s^\eta \tag{3.33}$$

in such a way that T_s^0 has half-integer eigenvalues. Defining $q_a^{(1)} = q_a$ for $a = 1 \dots n_1$ and $q_a^{(2)} = q_{a+n_1}$ for $a = 1 \dots n_2$, we can choose some value η, for which the charges satisfy $q_a^{(1)} + \eta \in \mathbb{Z} + 1/2$ and $q_a^{(2)} - \eta \in \mathbb{Z} + 1/2$, i.e. T_s^η has eigenvalues $-\eta$ and η respectively. There is no need for the q_a's to be integers or half integers. We use the decomposition

$$\chi_\pm^a = \frac{1}{\sqrt{C}} \times \begin{cases} e^{i(q_a \pm l)\theta} U_\pm^a(r) & a = 1 \dots n_1 \\ e^{i(q_a \mp l)\theta} V_\pm^{a*}(r) & a = (n_f - n_2 + 1) \dots n_f \end{cases} \tag{3.34}$$

instead of eq. (3.14). We now impose that $\chi_\pm^a$ is single-valued at the origin implying that $q^a \pm l \in \mathbb{Z} + \frac{1}{2}$. This implies that $l \mp \eta \in \mathbb{Z}$.

Near the origin U^a and V^a satisfy eqs. (3.16) and (3.17), although with a restricted choice of a. The leading order behaviour of the solutions is given by eq. (3.18), and so the total number of normalisable solutions there is

$$N_0^\pm(l) = \sum_{a=1}^{n_1} I\left(l \geq \mp q_a^{(1)} + \frac{1}{2}\right) + \sum_{a=1}^{n_2} I\left(l \leq \pm q_a^{(2)} - \frac{1}{2}\right). \tag{3.35}$$

The behaviour of the spinors at infinity is determined by eqs. (3.20) and (3.21). We define n_{z1} and n_{z2} to be the number of massless fermions with respectively $a \leq n_1$ and $a > n_1$. The number of massive fermions is then $2\bar{n}$, where $\bar{n} = n_1 - n_{z1} = n_2 - n_{z2}$. At large r the approximate solutions are either exponential or are given by eq. (3.22). The total number of normalisable $\chi_\pm^a$ solutions is

$$N_\infty^\pm(l) = \bar{n} + n_{z1} I\left(l < -\frac{C_1}{2}\right) + n_{z2} I\left(l > \frac{C_1}{2}\right). \tag{3.36}$$

Using the fact that $l \mp \eta \in \mathbb{Z}$, we can express $N_\infty^\pm(l)$ in a similar form to eq. (3.35) by introducing

$$\tilde{q}_\pm^{(1)} = \pm\frac{1}{2} - \eta \mp \left[\frac{C_1}{2} \mp \eta\right] \tag{3.37}$$

and

$$\tilde{q}_\pm^{(2)} = \pm\frac{1}{2} + \eta \mp \left[\frac{C_1}{2} \pm \eta\right] \tag{3.38}$$

where $[x]$ is the lowest integer which is strictly greater than x. If $2\eta \in \mathbb{Z}$, then the above expressions agree with eq. (3.24), and $\tilde{q}_\pm^{(1)} = \tilde{q}_\pm^{(2)}$.

We can now write

$$N_\infty^\pm(l) = \bar{n} + n_{z2} I\left(l \geq \mp\tilde{q}_\pm^{(1)} + \frac{1}{2}\right) + n_{z1} I\left(l \leq \pm\tilde{q}_\pm^{(2)} - \frac{1}{2}\right), \tag{3.39}$$

and proceed in a similar way to the previous subsection. We define $\hat{q}_a^{(1)}$ to be the combination of the $q_a^{(1)}$ and n_{z2} copies of $\tilde{q}_\pm^{(1)}$, and order them so that $\hat{q}_1^{(1)} \leq \dots \hat{q}_{\bar{n}+n_z}^{(1)}$, where

$n_z = n_{z1} + n_{z2}$. Similarly we define $\hat{q}^{(2)}$ to be the $q_a^{(2)}$ and n_{z1} copies of $\tilde{q}_\pm^{(2)}$, and order them so that $\hat{q}_1^{(2)} \geq \ldots \hat{q}_{\bar{n}+n_z}^{(2)}$. This allows us to write

$$N_0^\pm(l) + N_\infty^\pm(l) = n_f$$
$$+ \sum_{a=1}^{\bar{n}+n_z} \left\{ I\left(\mp\hat{q}_a^{(1)} + \frac{1}{2} \leq l \leq \pm\hat{q}_a^{(2)} - \frac{1}{2}\right) - I\left(\pm\hat{q}_a^{(2)} - \frac{1}{2} < l < \mp\hat{q}_a^{(1)} + \frac{1}{2}\right)\right\}. \quad (3.40)$$

In contrast to the earlier case, solutions for different l are independent, and so the total number of real solutions is $N^\pm = 2\sum_l N^\pm(l)$. Again only half the terms in the above expression (3.40) are non-zero, which allows us to reorder the summations in the expressions for $N^\pm$, and obtain

$$N^\pm = 2\sum_{a=1}^{\bar{n}+n_z} \left[\pm\hat{q}_a^{(1)} \pm \hat{q}_a^{(2)}\right]_+ . \quad (3.41)$$

Note that there are different definitions of the $\tilde{q}_\pm^{(i)}$ in the $\hat{q}_a^{(i)}$ for the two chiralities. This last expression is particularly useful in supersymmetry where most models have block off-diagonal mass matrices.

It is interesting to note that the values of $\tilde{q}_\pm^{(1,2)}$ depend on the string deficit angle, and hence may change if the string mass per unit length changes (as may occur at a phase transition). If the mass matrix has zero eigenvalues, then the number of zero modes (which will depend on $\tilde{q}_\pm^{(1,2)}$) could also be altered, and so $N^\pm$ will be sensitive to changes in the string's gravity. This is not the case if all the fermion fields are massive away from the string.

3.4 Zero mass matrix

Finally, we will determine the number of zero modes when the mass matrix is zero. In fact $M_{ab} \equiv 0$ is a special case of a block off-diagonal matrix, although we will cover it separately. In this case the only things determining whether there are fermion zero modes are the gauge fields and gravity. We take $\chi_\pm^a = e^{i(q_a \mp l)\theta} V_a^{\pm *}(r)/\sqrt{C}$. The complex solution will be normalisable at $r = 0$ if $l \leq \pm q_a - 1/2$. At $r = \infty$ it is normalisable if $l > C_1/2$, or equivalently if $l \geq \mp\tilde{q}_\pm^{(1)} + 1/2$, with $\tilde{q}_\pm^{(1)}$ given by eq. (3.37). Hence

$$N^+ = 2\sum_{a=1}^{n_f} \left[q_a + \tilde{q}_+^{(1)}\right]_+ , \quad N^- = 2\sum_{a=1}^{n_f} \left[-q_a - \tilde{q}_-^{(1)}\right]_+ . \quad (3.42)$$

Let us apply this result to the case treated in ref. [21]. When $l \in \mathbb{Z}$, the compensating factor $\eta = 0$, and there is only one charge $q_a = Q > 0$, the number of zero modes is given by

$$N^+ = 2\left[Q - \frac{1}{2}\right]_+ , \qquad N^- = 0 . \quad (3.43)$$

This is exactly equal to the number of integers satisfying $0 \leq m < Q - 1$ when $Q \in \mathbb{Z} + 1/2$.

4. Supersymmetry examples

In global supersymmetry, one can find zero modes around D- and F-term strings. As an application of the index theorem, we review some known examples and count the associated zero modes. We also consider the case where an F-term string couples to moduli fields as might be the case in string theory.

4.1 D-term strings

This model uses one superfield ϕ which is charged under an Abelian gauge group. The presence of a Fayet–Iliopoulos term ξ implies that the potential depends only on

$$D = -|\phi|^2 + \xi \,. \tag{4.1}$$

The cosmic string is such that ϕ interpolates between $\phi = 0$ and $\phi = \sqrt{\xi}$. Other fields may be present, but they will be zero everywhere for the string solution. The fermion mass matrix in global SUSY is off-diagonal and involves only χ and the gaugino λ. The winding numbers for the index theorem are $q_\chi^{(1)} = n$ and $q_\lambda^{(2)} = 0$. In supersymmetry, the number of zero modes is therefore given by [9]

$$N^+ = 2n \,, \qquad N^- = 0 \tag{4.2}$$

for $n > 0$. For negative n the two chiralities are interchanged, and there are $2|n|$ zero modes with negative chirality.

Using the analysis of the previous section, we can obtain the approximate form of the zero mode solutions. Taking $n > 0$, and defining $m = -l - 1/2$, we find that near the origin the positive chirality zero modes have the form

$$\lambda \sim r^m e^{i(m+1/2)\theta} \,, \qquad \chi \sim r^{n-1-m} e^{i(n-m-1/2)\theta} \,. \tag{4.3}$$

For the solution to be normalisable, the integer m must satisfy $0 \leq m \leq n-1$. These approximate solutions are obtained from eq. (3.18). For large r, λ and χ decay exponentially. As was shown in ref. [9], it is also possible to obtain the above $m = 0$ solution by applying a supersymmetry transformation to the string solution.

4.2 F-term strings

Consider the supersymmetric model with superpotential

$$W = \phi(\phi^+\phi^- - x^2) \tag{4.4}$$

and a U(1) gauge group. x sets the scale of the U(1) symmetry breaking. The fields $\phi^\pm$ have charges ± 1, and ϕ is uncharged. This model can be used to give hybrid inflation, with ϕ as the inflaton. In that case inflation ends with the breaking of the U(1) symmetry, which results in the production of cosmic strings. These have the form

$$\phi^\pm = x f(r) e^{\pm in\theta} \,, \qquad \phi = 0 \,. \tag{4.5}$$

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The mass matrix of the fermions contains the Yukawa and gauge interactions

$$-\left(\phi^-\chi - i\sqrt{2}\bar{\phi}^+\lambda\right)\chi^+ - \left(\phi^+\chi + i\sqrt{2}\bar{\phi}^-\lambda\right)\chi^- - \phi\chi^+\chi^- \tag{4.6}$$

where χ^i are the partners of the ϕ^i. Since $\phi = 0$ this mass matrix is block off-diagonal, and so we use the index theorem (3.41). The winding numbers are $q^{(1)}_{\chi+} = n$, $q^{(2)}_{\chi-} = -n$ and the other two q are zero. Hence we see that [9]

$$N^+ = 2|n|\,,\ N^- = 2|n|\,. \tag{4.7}$$

There are $|n|$ Weyl zero modes of each chirality.

If our toy model is extended, for example to include additional moduli fields, then it is possible that their coupling to ϕ will mean that ϕ is no longer zero inside the string. The mass matrix will then cease to have a block off-diagonal form, and the other index theorem (3.30) will apply. It tells us that now

$$N^+ = 0\,,\ N^- = 0\,. \tag{4.8}$$

Hence the zero modes are removed if the $\phi = 0$ solution is destabilised. A similar effect was noted in ref. [12] for SUSY breaking.

5. Fermion zero modes in supergravity

5.1 Fermions in supergravity

We will now extend our zero mode analysis to supergravity theories. This is complicated by the inclusion of the gravitino ψ_μ, whose mass and kinetic terms have a different form to those used in section 3. We keep the discussion general and look at cosmic strings with an arbitrary number of chiral superfields (ϕ^i, χ^i), and one U(1) gauge superfield (A_μ, λ). Working in Planck units, the fermion part of the supergravity Lagrangian is

$$\begin{aligned}\mathcal{L} = {}& \epsilon^{\mu\nu\rho\lambda}\bar{\psi}_\mu\bar{\sigma}_\nu D_\rho\psi_\lambda - i\bar{\lambda}\bar{\sigma}^\mu D_\mu\lambda - iK_{i\bar{j}}\bar{\chi}^{\bar{j}}\bar{\sigma}^\mu D_\mu\chi^i \\ & +\frac{1}{2}\bar{\psi}_\mu(D + i\bar{\sigma}^{\rho\lambda}F_{\rho\lambda})\bar{\sigma}^\mu\lambda - \frac{1}{\sqrt{2}}K_{i\bar{j}}D_\nu\phi^i\bar{\psi}_\mu\bar{\sigma}^\nu\sigma^\mu\bar{\chi}^{\bar{j}} + i\sqrt{2}K_{i\bar{j}}Q\phi^i\bar{\chi}^{\bar{j}}\bar{\lambda} \\ & -m_{3/2}\bar{\psi}_\mu\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu + \frac{i}{\sqrt{2}}m_i\bar{\psi}_\mu\bar{\sigma}^\mu\chi^i - \frac{1}{2}m_{ij}\chi^i\chi^j + \Lambda\sigma^\mu\bar{\psi}_\mu + \text{(c.c.)}\end{aligned} \tag{5.1}$$

where $\bar{\sigma}^{\mu\nu} = (\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)/4$. We have defined $m_{3/2} = e^{K/2}W$, $m_i = e^{K/2}D_iW$ and $m_{ij} = e^{K/2}D_iD_jW$ with $D_iW = \partial_iW + K_iW$.

In supergravity, the supersymmetry transformations preserve the Lagrangian, and act as gauge transformations. When analysing the theory it is important to fix the gauge. We do this by imposing

$$\bar{\sigma}^\mu\psi_\mu = 0\,. \tag{5.2}$$

To achieve this we have incorporated a Lagrange multiplier Λ, in the above Lagrangian. In general this does not fix the gauge symmetry fully. Residual gauge symmetries are still present corresponding to $\bar{\sigma}^\mu\delta\psi_\mu = 0$. In the following we will not fix them.

For the above theory, the fermion supersymmetry transformations are

$$\delta\chi^i = i\sqrt{2}\sigma^\mu\bar{\epsilon}D_\mu\phi^i - \sqrt{2}e^{K/2}K^{i\bar{\jmath}}D_{\bar{\jmath}}\bar{W}\epsilon \tag{5.3}$$

$$\delta\lambda = (F_{\mu\nu}\sigma^{\mu\nu} - iD)\epsilon\,. \tag{5.4}$$

The gravitino variation is

$$\delta\psi_\mu = 2\left(\partial_\mu + \omega_\mu + \frac{i}{2}A^B_\mu\right)\epsilon + im_{3/2}\sigma_\mu\bar{\epsilon} \tag{5.5}$$

where A^B_μ is defined in eq. (2.15).

In general, cosmic string solutions break all supersymmetries. However there exist BPS cosmic strings which preserve half of the original supersymmetries. These configurations are solutions of the Killing spinor equations obtained by equating some of the above fermion variations to zero. BPS configurations are of particular interest in string theory.

Let us now look for solutions of the Killing spinor equations in a cosmic string background, where the fields depend on r and θ only. Such solutions will be BPS cosmic strings. The t and z components of the gravitino variation lead to the equations

$$B'\sigma^1\epsilon = m_{3/2}\bar{\epsilon}\,, \qquad B'\sigma^1\epsilon = -m_{3/2}\bar{\epsilon} \tag{5.6}$$

whose only solution is obtained when both $m_{3/2} = 0$ and $B' = 0$. Hence a non-zero gravitino mass is not compatible with the existence of BPS states. The conformal factor e^{2B} must also be a constant.

Taking $\epsilon(r,\theta) = e^{i\sigma^3\theta/2}\epsilon_0(r)$, the variations of the other fields for a cosmic string background are

$$\delta\lambda = -i(F_{12}\sigma^3 + D)\epsilon \tag{5.7}$$

$$\delta\chi^i = \sqrt{2}i\left(\sigma^r\partial_r\phi^i + \sigma^\theta D_\theta\phi^i\right)\bar{\epsilon} \tag{5.8}$$

$$\delta\psi_\theta = i\left(\sigma^3 - 2iw_\theta + A^B_\theta\right)\epsilon\,, \quad \delta\psi_r = 2\partial_r\epsilon\,. \tag{5.9}$$

For a BPS cosmic string, these must all vanish for some choice of ϵ. This condition gives the required field equations for the string solution. We take $\epsilon_0(r)$ to be a constant and an eigenstate of σ^3. From this we obtain the BPS equations (2.12)–(2.14). Note that they are first order differential equations, and that 1/2 of the supersymmetry is preserved.

5.2 Gravitino field equations

Using the previous Lagrangian, the gravitino equations are given by

$$\epsilon^{\mu\nu\rho\lambda}\bar{\sigma}_\nu D_\rho\psi_\lambda + \frac{1}{2}(D + iF_{\rho\lambda}\bar{\sigma}^{\rho\lambda})\bar{\sigma}^\mu\lambda - \frac{1}{\sqrt{2}}K_{i\bar{\jmath}}D_\nu\phi^i\bar{\sigma}^\nu\sigma^\mu\bar{\chi}^{\bar{\jmath}}$$
$$-2m_{3/2}\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu + \frac{i}{\sqrt{2}}m_i\bar{\sigma}^\mu\chi^i = \bar{\sigma}^\mu\Lambda \tag{5.10}$$

and the spin 1/2 equations are

$$-i\bar{\sigma}^\mu D_\mu\lambda + i\sqrt{2}K_{i\bar{\jmath}}T^s\phi^i\bar{\chi}^{\bar{\jmath}} + \frac{1}{2}\bar{\sigma}^\mu(D + iF_{\rho\lambda}\sigma^{\rho\lambda})\psi_\mu = 0 \tag{5.11}$$

$$-i\bar{\sigma}^\mu D_\mu\chi^i + i\sqrt{2}T^s\phi^i\bar{\lambda} - \frac{1}{\sqrt{2}}D_\nu\phi^i\bar{\sigma}^\mu\sigma^\nu\bar{\psi}_\mu - K^{\bar{\jmath}i}\bar{m}_{\bar{\jmath}\bar{k}}\bar{\chi}^{\bar{k}} - \frac{i}{\sqrt{2}}K^{\bar{\jmath}i}\bar{m}_{\bar{\jmath}}\bar{\sigma}^\mu\psi_\mu = 0\,. \tag{5.12}$$

This set of equations can be mapped to the Dirac equations used in the proof of the index theorem. No manipulation is necessary for the spin 1/2 fermions. For the gravitino, more work is needed.

The fact that our cosmic string background is independent of z and t, simplifies the above equations. In particular we see that the connection terms in the covariant derivatives $D_{z,t} = \partial_{z,t} + w_{z,t}$ are

$$w_z = i\sigma^2 B'/2\,, \qquad w_t = -\sigma^1 B'/2 \tag{5.13}$$

which depend only on B'. We also have $w_r = 0$. The D_θ derivatives are

$$\begin{aligned} D_\theta \bar\psi_\mu &= \left(\partial_\theta - \frac{i\sigma^3}{2}C' - \frac{i}{2}A^B_\theta\right)\bar\psi_\mu\,,\ D_\theta\lambda = \left(\partial_\theta - \frac{i\sigma^3}{2}C' + \frac{i}{2}A^B_\theta\right)\lambda \\ D_\theta\chi^i &= \left(\partial_\theta - \frac{i\sigma^3}{2}C' - iQ_iA_\theta - \frac{i}{2}A^B_\theta\right)\chi^i + \Gamma^i_{jk}D_\theta\phi^j\chi^k \end{aligned} \tag{5.14}$$

where A^B_μ is given by eq. (2.15).

After eliminating the Lagrange multiplier, one obtains three equations which can be cast in Dirac-like form. After gauge fixing, the gravitino field has 3 components. For a cosmic string background it is convenient to write them in terms of the three independent Weyl fermions

$$\Sigma = \sigma^r\bar\psi_r + \sigma^\theta\bar\psi_\theta\,, \qquad \Psi = \sigma^r\bar\psi_r - \sigma^\theta\bar\psi_\theta\,, \qquad \Pi = \sigma^t\bar\psi_t - \sigma^z\bar\psi_z \tag{5.15}$$

The gravitino equations can then be expressed as

$$i(\sigma^r[\partial_r + B'] + \sigma^\theta D_\theta)\Pi + 2i\sigma^z w_z\Psi - m_{3/2}\bar\Pi - i(\bar\sigma^t\partial_t - \bar\sigma^z\partial_z)\Sigma = 0 \tag{5.16}$$

for the Π equation, and

$$i(\sigma^r[\partial_r + B'] - \sigma^\theta D_\theta)\Sigma - \sqrt{2}K_{\bar\imath j}[\sigma^r\partial_r - \sigma^\theta D_\theta]\bar\phi^{\bar\imath}\chi^j + m_{3/2}\bar\Psi + i(\bar\sigma^t\partial_t + \bar\sigma^z\partial_z)\Psi = 0 \tag{5.17}$$

for the Σ equation and a combination

$$\begin{aligned} &i[\sigma^r(\partial_r + C'/C) - \sigma^\theta\tilde D_\theta]\Psi - i[\sigma^r(\partial_r + C'/C) + \sigma^\theta\tilde D_\theta]\Sigma + 2i\sigma^z w_z\Pi + 2\sigma^3F_{12}\bar\lambda \\ &\qquad -\sqrt{2}K_{\bar\imath j}\sigma^\nu D_\nu\bar\phi^{\bar\imath}\chi^j + 2m_{3/2}\bar\Sigma + i(\bar\sigma^t\partial_t - \bar\sigma^z\partial_z)\Pi + i(\bar\sigma^t\partial_t + \bar\sigma^z\partial_z)\Sigma = 0 \end{aligned} \tag{5.18}$$

closing the system. We have defined $\tilde D_\theta = \partial_\theta + i\sigma^3C'/2 - iA^B_\theta/2$. These equations are coupled in general.

In a similar manner the spin 1/2 equations are given by

$$i(\sigma^r\partial_r + \sigma^\theta D_\theta)\lambda + i\sqrt{2}K_{i\bar\jmath}T^s\phi^i\bar\chi^{\bar\jmath} - \sigma^3F_{12}\bar\Sigma - i(\bar\sigma^t\partial_t + \bar\sigma^z\partial_z)\lambda = 0 \tag{5.19}$$

and

$$\begin{aligned} &i(\sigma^r D_r + \sigma^\theta D_\theta)\chi^i + i\sqrt{2}T^s\phi^i\bar\lambda - K^{\bar\jmath i}\bar m_{\bar\jmath\bar k}\bar\chi^{\bar k} \\ &+\frac{1}{\sqrt{2}}\sigma^\nu D_\nu\phi^i\Sigma + \frac{1}{\sqrt{2}}[\sigma^r\partial_r - \sigma^\theta D_\theta]\phi^i\Psi - i(\bar\sigma^t\partial_t + \bar\sigma^z\partial_z)\chi^i = 0\,. \end{aligned} \tag{5.20}$$

The zero mode solutions we are looking for depend on r and θ, and so we can ignore the z and t derivatives. For fermion currents on the string (which are physically more interesting), the z and t dependence of the fields will have to be included. We will discuss this in section 7.

Let us now discuss the normalisation of the gravitino fields. We choose a norm compatible with the requirements of canonical quantisation in field theory

$$||\psi_\mu||^2 = \frac{i}{2}\int dr d\theta e^{2B} C \left[\Sigma P_\Sigma + \Pi P_\Pi + \Psi P_\Psi - (\text{c.c.})\right] . \tag{5.21}$$

We identify $P_X = \delta\mathcal{L}/(\delta\partial_0 X)$ as the conjugate momentum, where $\mathcal{L}$ is the Lagrangian (5.1). The above norm for the gravitino part of the wavefunction is equal to

$$||\psi_\mu||^2 = \frac{1}{2}\int dr d\theta C e^{2B} \bar{\psi}_\mu \eta^{\mu\nu} \psi_\nu . \tag{5.22}$$

The full norm for the fermion states is then

$$\int dr d\theta C e^{2B} \left[\sum_i |\chi^i|^2 + \frac{1}{2}\left(|\Psi|^2 + |\Sigma + \Pi|^2 - |\Pi|^2\right)\right] . \tag{5.23}$$

This must be finite if the zero mode solution is to be a normalisable bound state. Note that the positivity of the norm is not guaranteed. However since we expect the string to be stable, any normalisable fermion bound states should have non-negative norms. It is still possible that they are gauge artifacts, and so there is a risk that our analysis leads to some unphysical zero modes. If we had been working in Minkowski space (instead of a string background) we could have removed any unphysical modes with further gauge fixing. It is not at all clear how to do this in a string background and we leave the possibility of using residual supersymmetries to further work.

5.3 Index theorem

We see that the Dirac-like equations for the gravitino fields (5.16)–(5.18) have a different form to that assumed in eqs. (3.10), (3.11), which were used to derive the number of fermion zero modes. The expressions derived in section 3 cannot therefore be applied directly to models which include gravitinos. However, the number zero modes can still be determined by analysing approximate large and small r solutions. As we will show, this does allow our index theorem to be modified to include gravitinos.

We will start by extending the index theorem for block off-diagonal mass matrices (3.41), as this will be the relevant case for D-strings. In this case the gravitino mass term $m_{3/2}$ must be zero. For large and small r the gravitino Dirac equations reduce to

$$\left(\partial_r \mp \frac{i}{C}\left[\partial_\theta \mp \frac{i}{2}C' + iq_\psi a\right]\right)\Sigma_\pm = 0 \tag{5.24}$$

$$\left(\partial_r + \frac{C'}{C} \mp \frac{i}{C}\left[\partial_\theta \pm \frac{i}{2}C' + iq_\psi a\right]\right)\Psi_\pm \propto \frac{i}{C}\partial_\theta \Sigma_\pm ,\ B'\Pi_\pm \tag{5.25}$$

$$\left(\partial_r \pm \frac{i}{C}\left[\partial_\theta \mp \frac{i}{2}C' + iq_\psi a\right]\right)\Pi_\pm \propto B'\Psi_\pm \tag{5.26}$$

where we have defined $q_\psi = -n\xi/2$. We have neglected $A^B_\mu - \xi A_\mu$, $D_\mu\phi^i$, and $\sigma^3 F_{12}$ (which would form part of M_{ab} and J_μ in eq. (3.2)), since they are all subdominant as r tends to 0 or ∞.

These equations are analogous to a Dirac equation and can be analysed both at the origin and at infinity. Following the ansatz (3.34) we take

$$\Psi = \frac{V^\Psi_\pm}{\sqrt{C}} e^{-i(q_\psi \mp l)\theta}\,, \qquad \Sigma = \frac{V^\Sigma_\pm}{\sqrt{C}} e^{-i(q_\psi \mp l)\theta}\,, \qquad \Pi = \frac{U^\Pi_\pm}{\sqrt{C}} e^{i(-q_\psi \pm l)\theta}\,. \tag{5.27}$$

As in section 3, we look for the leading order behaviour of the fermion equations near the origin. The gauge field A_θ is negligible there, so we can ignore it in eqs. (5.24)–(5.26).

We start with a solution whose leading order behaviour comes from Σ. This is obtained by solving eq. (5.24). Eq. (5.25) implies Ψ will have similar behaviour. The behaviour of other fields comes from their coupling to Σ and Ψ via Yukawa terms. Since the string solution is regular at the origin, we find the other fermion fields are at least as well behaved as Σ there. Putting this all together gives the leading order behaviour of the solution

$$V^\Sigma_\pm, V^\Psi_\pm \sim r^{\pm q_\psi + 1 - l}\,, \qquad \chi^i, \lambda, \Pi = O(r^{\pm q_\psi - l + 2})\,. \tag{5.28}$$

Another solution is obtained by taking the right-hand side of eq. (5.25) to be subdominant, and then solving for Ψ. The leading behaviour of the other fields then comes from their coupling to Ψ via Yukawa terms. Hence

$$V^\Psi_\pm \sim r^{\pm q_\psi - 1 - l}\,, \qquad \chi^i, \lambda, \Sigma, \Pi = O(r^{\pm q_\psi - l})\,. \tag{5.29}$$

Finally, we take the right-hand side of eq. (5.26) to be subdominant, and solve for Π to obtain the third independent solution. Substituting this expression for Π into eq. (5.25), and using $B' \sim r$ near the origin, we can obtain the leading behaviour of Ψ and Σ. As with the other two solutions, the behaviour of the non-gravitino fields is determined by their couplings to Yukawa terms. We find

$$U^\Pi_\pm \sim r^{\mp q_\psi + l}\,, \qquad V^\Sigma_\pm, V^\Psi_\pm \sim r^{\mp q_\psi + l + 2}\,, \qquad \chi^i, \lambda = O(r^{\mp q_\psi + l + 3})\,. \tag{5.30}$$

Hence we need respectively $l \leq \pm q_\psi - 3/2$, $l \leq \pm q_\psi + 1/2$ and $l \geq \mp q_\psi - 1/2$ for each of the above three solutions to be normalisable inside the string core. In order to use the same algebra that leads to the expression (3.41), we want to write the above conditions in the same form as eq. (3.35). This is achieved by introducing the chirality dependent effective charges

$$q^{(2)}_\Psi = q_\psi \mp 1\,, \qquad q^{(2)}_\Sigma = q_\psi \pm 1\,, \qquad q^{(1)}_\Pi = -q_\psi \pm 1\,. \tag{5.31}$$

Notice the shifts of one unit.

We also need to consider the behaviour of the other fermion solutions, whose leading behaviour comes from χ^i and λ. For these solutions the gravitino fields are subdominant, and so the analysis of section 3 still applies.

The gravitino is massless at infinity and so, as was discussed in section 3, some zero modes with low angular momentum will be lost. We can again account for this by introducing some effective winding numbers $\tilde{q}_\pm$. For the gravitino fields we find the leading order behaviour of three approximate solutions at infinity is

$$V_\pm^\Psi \sim r^{-1-l/C_1} \tag{5.32}$$
$$V_\pm^\Sigma \sim V_\pm^\Psi \sim r^{1-l/C_1} \tag{5.33}$$
$$U_\pm^\Pi \sim r^{l/C_1}\,, \qquad V_\pm^\Sigma, V_\pm^\Psi \sim r^{l/C_1} \tag{5.34}$$

where we have used $B' \sim 1/r$ far away from the string. Following the reasoning behind eq. (3.37), we see that the above solutions are normalisable if $l \geq \mp\tilde{q}^{(1)}_{\Psi\pm}+1/2$, $l \geq \mp\tilde{q}^{(1)}_{\Sigma\pm}+1/2$ and $l \leq \pm\tilde{q}^{(2)}_\pm - 1/2$ with

$$\tilde{q}^{(1)}_{\Psi\pm} = \pm\frac{1}{2} - \eta \mp \left[-\frac{C_1}{2} \mp \eta\right] \tag{5.35}$$
$$\tilde{q}^{(1)}_{\Sigma\pm} = \pm\frac{1}{2} - \eta \mp \left[\frac{3C_1}{2} \mp \eta\right] \tag{5.36}$$

and $\tilde{q}^{(2)}_\pm$ is defined in eq. (3.37). The single-valuedness of the gravitino implies that we take $\eta = (1-n\xi)/2$. Using $0 < C_1 \leq 1 - |n|\xi$, we find that

$$\tilde{q}^{(1)}_{\Psi\pm} = \frac{n\xi}{2} = -q_\psi \tag{5.37}$$
$$\tilde{q}^{(1)}_{\Sigma\pm} = \frac{n\xi}{2} \mp I\left(3[1-C_1] \leq 2 \pm n\xi\right) \tag{5.38}$$
$$\tilde{q}^{(2)}_\pm = -\frac{n\xi}{2} \mp I\left(1 - C_1 = \mp n\xi\right) \tag{5.39}$$

hence $\tilde{q}^{(1)}_{\Sigma\pm} = -q^{(2)}_\Sigma$ unless the strings have a large deficit angle (which implies they must be very heavy). For non-BPS strings, we always have $\tilde{q}^{(2)}_\pm = q_\psi$.

With the aid of the modified charges $q^{(2)}_{\Sigma,\Psi}$ and $\tilde{q}^{(1)}_{\Psi,\Sigma\pm}$, we can now use the index theorem (3.41) to obtain the total number of zero modes, even when gravitinos are included in the model.

If the fermion mass matrix is more generic, i.e. is not block off-diagonal, we must instead use the ansatz

$$\Psi = \frac{e^{-iq_\psi\theta}}{\sqrt{C}}\left\{U_\pm^{\Psi*}e^{\mp il\theta} + V_\pm^\Psi e^{\pm il\theta}\right\} \tag{5.40}$$
$$\Sigma = \frac{e^{-iq_\psi\theta}}{\sqrt{C}}\left\{U_\pm^{\Sigma*}e^{\mp il\theta} + V_\pm^\Sigma e^{\pm il\theta}\right\} \tag{5.41}$$
$$\Pi = \frac{e^{-iq_\psi\theta}}{\sqrt{C}}\left\{U_\pm^{\Pi}e^{\pm il\theta} + V_\pm^{\Pi*} e^{\mp il\theta}\right\}. \tag{5.42}$$

The single-valuedness of the wave functions implies that q_ψ must be integer or half integer.

As before we are interested in the leading order behaviour of the above solutions at the origin. Using the effective charges defined above, it is given by eq. (3.18). If $m_{3/2} \neq 0$, then the solutions will have exponential behaviour at infinity, and the analysis of subsection 3.2 will apply. If on the other hand $m_{3/2} = 0$, then some of the large r gravitino solutions will have power law behaviour. As before, we can deal with this by including some extra

$\tilde{q}_\pm$ winding numbers. Instead of the expression (3.24), we use the above effective winding numbers (5.35)–(5.36). With all these modifications, the index theorem (3.30) will now apply to models with gravitinos.

The expressions (3.41) and (3.30) now give the number of fermion bound states with finite norm. However if the field Π is non-zero we cannot be certain that the norm is positive, and so some of these solutions may be gauge artifacts or even ghosts. If the wavefunctions are dominated by the other fields, the norm will be positive, so it is only solutions whose leading order behaviour comes from Π which are likely to be gauge artifacts. This suggests that we may be able to avoid counting the gauge zero modes by excluding the winding numbers corresponding to Π from the analysis. However this is not guaranteed to work, and so we will consider both possibilities in what follows.

6. Supergravity examples

We will now apply our index theorem to cosmic strings in supergravity models. In contrast to the models discussed in section 4, we must now include the gravitino field.

6.1 D-strings

For the global SUSY equivalent of the D-string (see subsection 4.1), it was possible to find a fermion bound state using the broken supersymmetry transformation. The same idea can be tried for D-strings. The transformations for a string background are given by eqs. (5.7)–(5.9). For $n > 0$, the Killing spinor $e^{-i\theta/2}\epsilon_{0-}$ corresponds to the preserved 1/2 supersymmetry. In supergravity, the spinor $e^{i\theta/2}\epsilon_{0+}(r)$ is the Goldstino of broken supersymmetry obtained in ref. [17]. In order to satisfy the gauge choice (5.2) we need $\epsilon_0(r) \propto \exp[\int(1-C')/C dr]$. From eq. (5.9) we obtain

$$\bar{\Psi} = \bar{\sigma}^r\psi_r - \bar{\sigma}^\theta\psi_\theta = 4\frac{1-C'}{C}\sigma^1\epsilon_{0+}(r)\,. \tag{6.1}$$

Hence $\Psi \propto n\xi r^{-2+1/C_1}$ at infinity, and it is therefore not a normalisable bound state. This contrasts with the situation for global supersymmetry, where the corresponding state is localised on the string. The disappearance of this zero mode can be confirmed using the index theorem.

Since we have a BPS string, the $m_{3/2} = 0$ and the B' terms in the gravitino equations (5.16)–(5.17) vanish. We see that Π decouples from the other fields. It is pure gauge so we can ignore it.

Taking $n > 0$, we see that the field Σ also decouples for positive chirality states. Solving eq. (5.17) we find $\Sigma = 0$. The fermion mass matrix is block off-diagonal, and so we separate the fields into two groups. The first contains the Higgsino with winding number $q^{(1)}_\chi = n - q_\psi$, and the second contains the gaugino ($q^{(2)}_\lambda = q_\psi$), and the gravitino field Ψ ($q^{(2)}_{\Psi\pm} = q_\psi \mp 1$). Since the gravitino field is massless at infinity we need to add the effective winding $\tilde{q}^{(1)}_{\Psi\pm} = -q_\psi$. For the negative chirality states Σ does not decouple, and so in this case so we also need to include $q^{(2)}_{\Sigma-}$ and $\tilde{q}^{(1)}_{\Sigma-}$ in the analysis. Using the index theorem (3.41) we find that (for $n > 0$) the number of zero modes is

$$N^+ = 2(n-1)\,, \qquad N^- = 0\,. \tag{6.2}$$

Comparing this with the results for global SUSY (4.2), we see that in the broken $N = 1/2$ sector corresponding to the positive chirality, a pair of zero modes, i.e. a complex spinor, has disappeared. In ref. [17] this result was suggested using a superHiggs argument, i.e. the Goldstino field does not lead to zero modes anymore as it is eaten by the gravitino. Our calculations justify this heuristic reasoning. This result is remarkable as it exemplifies the role of supergravity in the counting of zero modes.

Let us be more explicit and show which zero mode disappears in supergravity. It is helpful to introduce $m = -l - 1/2 + q_\psi$, which is an integer. Near $r = 0$, the general solution to the fermion field equations has the leading order behaviour

$$U^\chi = c_1 r^{n-m-1/2}\,, \qquad V^\lambda = c_2 r^{m+1/2}\,, \qquad V^\Psi = c_3 r^{m-1/2} \tag{6.3}$$

where the c_i are constants. At infinity one combination of solutions decays exponentially, and (for $m \geq 0$) is the only normalisable solution there. In general, the form of this combination of solutions near $r = 0$ will be given by the above expression with all $c_i \neq 0$. Hence if the solution is to be normalisable everywhere, we must have $m \leq n-1$, $m \geq 0$ and $m \geq 1$ (the last condition comes from the gravitino field). Without the gravitino we would need $n - 1 \geq m \geq 0$. Including it we lose the $m = 0$ mode as it cannot be normalisable close to the origin. The rest of the zero mode tower is preserved.

6.2 Non-BPS D-strings

We now suppose that the above strings are no longer BPS and that $B' \neq 0$. These strings have the same field content as BPS ones, but T^r_r is no longer zero. We see that the fields Π and Σ no longer decouple, and the analysis is changed. If we exclude the q corresponding to Π, we obtain almost the same result as for the BPS D-strings (for $n > 0$)

$$N^+ = 2(n-1) + 2I\left(3[1 - C_1] > 2 + n\xi\right), \qquad N^- = 0\,. \tag{6.4}$$

The $I(\ldots)$ term will be zero unless the string deficit angle is very big. If this is not the case the results of the previous section hold, although the fermion wavefunction will now include small contributions from the Π and Σ fields.

On the other hand if we do include the additional charges $q^{(1)}_\Pi = -q_\psi \pm 1$ and $\tilde{q}^{(2)}_\pm = q_\psi$ for the field Π, the index N^+ is increased by two. These extra zero modes do satisfy all the requirements for normalisability, although we cannot be sure that they have positive norm, and they may just be gauge. The corresponding solutions are gauge for BPS D-strings, suggesting they will also be gauge in this case, and should be ignored.

If the strings are very heavy, and $3(1 - C_1) > 2 + n\xi$, there will be an additional $m = -1$ zero mode. Near the origin the leading order behaviour of the solutions is

$$U^\chi = c_1 r^{n-m-1/2}\,, \qquad V^\Psi \sim V^\Sigma = c_2 r^{m+3/2}\,, \qquad U^\Pi = c_3 r^{-m-1/2} \tag{6.5}$$

and near infinity it is

$$V^{\lambda} \sim U^{\chi} = c_4 e^{-\sqrt{2}\xi}\,, \qquad V^{\Sigma} \sim r^{(m+1/2-q_\psi)/C_1+1}\,, \qquad V^{\Psi} = c_5 r^{(m+1/2-q_\psi)/C_1-1} + c_6 V^{\Sigma}\,. \tag{6.6}$$

Hence if $m = -1$ and $3C_1 < 1 - n\xi$, there is a solution to the fermion field equations which has the above form and is normalisable everywhere. We see here that the fermion wavefunctions have power law decay outside the string. However the above normalisable solution is only possible if the string deficit angle is very big, and such heavy strings are ruled out by astrophysical observations.

6.3 D-strings with spectators

We add to the D-strings a new field Φ and a superpotential

$$W = \frac{a}{2}\phi\Phi^2\,. \tag{6.7}$$

The fermion χ_Φ associated with Φ has winding $q_\Phi = -n/2$ as its charge is $Q_\Phi = -1/2$. In the string background, we have $\Phi = 0$, and the derivatives $\partial W/\partial\phi^i$ vanish too.

The field χ_Φ does not couple to any of the other fermions, and so it can be analysed independently. Its presence does not affect the results of the previous subsections. Applying the index theorem (with $n > 0$) we find

$$N^+ = 0\,, \qquad N^- = n\,. \tag{6.8}$$

The presence of these chiral zero modes has been obtained in ref. [22]. The above result also holds for global SUSY, and is not affected by the inclusion of SUGRA effects.

6.4 Massless spectators on D-strings

It is also possible for fermion fields which do not couple to a Higgs field to have zero modes on a D-string. In contrast to fermions which are massive off the string, the number of zero mode solutions is sensitive to the deficit angle of the string, which itself depends on whether the string is BPS or not.

Consider a fermion with winding $q = q_\psi + Qn$, where $Q > 0$ and Qn is an integer. The expression (3.42) will give the number of zero modes in this case. Taking $n > 0$, we have $\tilde{q}_+^{(1)} = -q_\psi - 1$ and $\tilde{q}_-^{(1)} = -q_\psi$ for a BPS string. Hence there are $nQ - 1$ positive chirality zero modes. On the other hand for a non-BPS string, the deficit angle is increased, which alters the value of $\tilde{q}_+^{(1)}$ to $-q_\psi$. This results in an additional zero mode, and so the conductivity of the string is actually increased by supersymmetry breaking. If we have $q = -q_\psi + Qn$ instead, the equivalent calculation gives Qn positive chirality zero modes, irrespective of whether the string is BPS or not.

Note that these zero modes have power law decay outside the string core, in contrast to the usual exponential decay. This implies corresponding currents will be less effective at stabilising vortons. The radius of a typical vorton is about ten times the size of the string core. If the zero mode wavefunctions decay exponentially outside the string core, the overlap between the wavefunctions of fermion states on opposite sides of the string loop will be tiny. On the other hand if the wavefunctions have power law decay, this overlap will be large, and we expect the fermions to scatter off each other, producing particles which are not confined to the string. Hence the current which was stabilising the loop will decay, and the vorton will collapse. However, we expect that some loops with very large radii will still be stable, but only a small number of such vortons will be produced by a string network. The corresponding vorton constraints are therefore much weaker than usual [16].

6.5 Non-BPS F-term strings

For non-BPS strings the three gravitino degrees of freedom Ψ, Σ and Π do not decouple, and we need to include the corresponding effective winding numbers defined in eq. (5.31). For non-zero $m_{3/2}$ the mass matrix is not block off-diagonal, and we must use the index theorem for generic mass matrices (3.30). The winding numbers, q_a of the fermions must then be all either half integer or integer.

As an example, we extend the F-term string model of subsection 4.2. This has no Fayet-Iliopoulos terms, and the three effective gravitino charges (5.31) reduce to ∓ 1, ± 1 and ± 1. As before the Higgsinos have charges n, $-n$, 0 and the gaugino is also chargeless. Since we are now using a generic mass matrix, the different chirality Higgsino zero modes mix together to form massive states, and the corresponding zero modes disappear. We discussed a similar effect for the SUSY F-term model, although in this case the mixing arose from an extra non-trivial Yukawa term. This could also occur for our SUGRA model (perhaps due to the effects of other sectors of the theory), although the gravitino coupling alone is enough to remove the zero modes. Hence the inclusion of gravitinos means that none of the global SUSY zero modes survive.

Using the index theorem we find that $N^+ = N^- = 2$. These extra zero modes arise from the inclusion of the Π gravitino field in the analysis. The fact that the norm is not positive definite suggests these may be gauge degrees of freedom. This appeared to be the case with the corresponding modes for the non-BPS D-strings, and so we suspect it is also the case here. If we remove the q arising from Π, we obtain $N^+ = N^- = 0$, and so there are no fermion zero modes.

7. Fermion currents

7.1 Extending zero modes to massless currents

The zero mode solutions we have found depend only on r and θ. We will show that for theories without gravitinos these can easily be extended to massless currents by adding a z and t dependent phase to the zero mode solution.

We see from the field equations (5.19) and (5.20) that if we ensure

$$(\bar{\sigma}^t \partial_t + \bar{\sigma}^z \partial_z)\lambda = (\bar{\sigma}^t \partial_t + \bar{\sigma}^z \partial_z)\chi^i = 0 \tag{7.1}$$

then the r and θ dependence of our zero mode solutions will still satisfy the equations of motion. Let us make the changes $U_\pm \to f_\pm(z,t)U_\pm$ and $V_\pm \to f_\pm(z,t)V_\pm$ to the fermion ansatz (3.14). Taking $f_\pm(z,t) = e^{i\omega(t \mp z)}$, the above conditions hold, and so this extension of our fermion zero mode solution also satisfies the field equations. We see that it has definite chirality (like the zero mode), and that it moves at the speed of light along the string. The direction of the current is determined by its chirality.

Let us now consider what happens in the presence of the gravitino field. Extending the above arguments to the zero modes on a BPS D-string (see subsection 6.1), we try the ansatz

$$\chi_{\pm}(x^{\mu}) = f_{\pm}(z,t)\chi_{\pm}(r,\theta)\,, \qquad \lambda_{\pm}(x^{\mu}) = f^{*}_{\pm}(z,t)\lambda_{\pm}(r,\theta)\,, \qquad \Psi_{\pm}(x^{\mu}) = f_{\pm}(z,t)\Psi_{\pm}(r,\theta)\,. \tag{7.2}$$

Since $(\bar{\sigma}^t\partial_t + \bar{\sigma}^z\partial_z)\Psi = 0$, the ansatz works, and we again have a massless current with definite chirality.

However if we try to include the other gravitino fields Σ and Π (as would be required for non-BPS strings), we find that the above extension of zero modes to currents runs into major difficulties. First of all we see that both $(\bar{\sigma}^t\partial_t + \bar{\sigma}^z\partial_z)\Sigma$ and $(\bar{\sigma}^t\partial_t - \bar{\sigma}^z\partial_z)\Sigma$ appear in the gravitino equations (5.18), (5.16). Both constraints cannot be met simultaneously unless $\Sigma = 0$. This is only consistent if the scalars satisfy

$$\sigma^{\mu}D_{\mu}\phi^i = 0 \tag{7.3}$$

and

$$m_{3/2} = 0\,. \tag{7.4}$$

If $B' \neq 0$, then in order to satisfy the gravitino equation (5.16), Π needs to have the same phase as Ψ, and so

$$\Pi_{\pm}(x^{\mu}) = f_{\pm}(z,t)\Pi_{\pm}(r,\theta)\,. \tag{7.5}$$

Now this is not generally consistent with eq. (5.18) as $(\bar{\sigma}^t\partial_t - \bar{\sigma}^z\partial_z)\Pi$ does not vanish for non-zero Π. This implies that

$$\Pi = 0 \tag{7.6}$$

and so for consistency we also need

$$B' = 0\,. \tag{7.7}$$

In conclusion, we find that the usual extension of zero mode solutions to massless currents is only consistent in supergravity for BPS backgrounds. In this case the only part of the gravitino which does not vanish is Ψ. In the non-BPS cases, the spectrum does not appear to contain any massless excitations, even though it does have zero modes. A possible resolution of this is that the normalisable zero modes must be extended to currents with a non-standard dispersion relation $k_t = h(k_z)$, instead of the usual $k_t = \pm k_z$. At low energy, for small k_z, this suggests that $k_t \approx h_1 k_z$ where $|h_1| \neq 1$. In the following subsection we find further indications that this might be the case.

7.2 Two dimensional effective action

We will now investigate the nature of the fermion currents on the string by considering the two-dimensional effective action there. We will just consider the low energy behaviour of fermionic excitations around the cosmic string background (so we ignore all fermion states apart from the zero modes and their corresponding currents).

Starting with the BPS case (so $\Pi = \Sigma = 0$) we substitute the ansatz (7.2) into the fermion action. We keep $f_\pm(z,t)$ as arbitrary functions, and take $\Psi_\pm(r,\theta)$, etc. to be one of our zero mode solutions. We obtain

$$-i\int dtdz(||\psi_\mu||^2 + |\chi|^2 + |\lambda|^2)\bar{f}\bar{\sigma}^i\partial_i f \tag{7.8}$$

where $i = z, t$ and

$$||\psi_\mu||^2 = \frac{1}{2}\int drd\theta C|\Psi|^2 . \tag{7.9}$$

Hence we have obtained an effective two-dimension action for a massless fermion, f. Each BPS normalisable zero mode leads to a massless current on the string world sheet. The norm used for the gravitino is positive definite and coincides with the one used in the index theorem (5.23).

Let us repeat the same analysis in the non-BPS D-strings. We have seen that the four-dimensional equations of motion do not lead to massless currents. Here we investigate the breakdown of the masslessness condition at the level of the effective action by extending the ansatz (7.2) to include

$$\Sigma_\pm(x^\mu) = f_\pm(z,t)\Sigma_\pm(r,\theta) , \qquad \Pi_\pm(x^\mu) = f_\pm(z,t)\Pi_\pm(r,\theta) . \tag{7.10}$$

Substituting this into the action gives

$$-i\int dtdz\bar{f}N^i\partial_i f \tag{7.11}$$

where

$$N^t = \bar{\sigma}^t\int drd\theta e^{2B}C\left[|\chi|^2 + |\lambda|^2 + \frac{1}{2}\left(|\Psi|^2 + |\Sigma + \Pi|^2 - |\Pi|^2\right)\right] \tag{7.12}$$

and

$$N^z = \bar{\sigma}^z\int drd\theta e^{2B}C\left[|\chi|^2 + |\lambda|^2 + \frac{1}{2}\left(|\Psi|^2 + |\Sigma - \Pi|^2 - |\Pi|^2\right)\right] . \tag{7.13}$$

As $N^t \neq N^z$, we find that the action does not describe a two-dimensional massless fermion. More precisely we find that the dispersion relation for the fermion $f_\pm$ becomes

$$k_t = \pm\frac{N^z}{N^t}k_z . \tag{7.14}$$

It would be very interesting to investigate the non-BPS zero modes and their associated currents further. Indeed, the non-BPS case seems to be plagued with two types of inconsistencies. First the zero modes are not guaranteed to have a positive norm, although residual gauge symmetries might be enough to gauge away the would-be negative norm states. Even if the zero modes are of positive norm, their physical relevance is not at all clear. Indeed they do not seem to be naturally extendible to massless currents, and seem to have two-dimensional Lorentz breaking dispersion relations. It could be that the normalisable zero modes in the non-BPS case are all gauge artifacts. This would guarantee the absence of two-dimensional Lorentz invariance breaking currents. However this seems unlikely, as a BPS D-string (which has problem-free massless states) can be obtained as a limiting case of a non-BPS string. More work is needed in this direction in order unravel the physics of zero modes in the non-BPS case.

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8. Conclusion

We have derived a general expression for the number of fermion zero modes bound to a cosmic string. This index theorem is valid for a general model, even if it includes massless fermions and gravitinos (both possibilities being important for supergravity). The presence of massless eigenstates in the mass matrix reduces the number of zero modes. Physically this can be interpreted as the bound states mixing with free massless states. The presence of gravitinos also reduces the number of zero modes, firstly because the gravitinos are massless away from the string and secondly because the string confines them less effectively due to their different kinetic terms. Since strong constraints on conducting strings arise from the presence of vortons, reduced conductivity of supergravity strings implies that the vorton constraints will be relaxed.

In particular for D-term strings in supergravity, we find there is one less zero mode state than the corresponding model without gravitinos. This implies that there is no zero mode on a cosmic string with winding $n = 1$. This is consistent with the results of ref. [17]. Consequently these D-strings evade the stringent constraints on chiral vortons which their global analogues were subject to. Higher winding number strings still have zero modes. It is interesting to ask what happens to states on a $n > 1$ string if it splits into several $n = 1$ strings. Does the wavefunction spread out over the different strings, or must it decay?

There are rather different physical reasons behind the reduction in zero modes in the D-term and F-term cases. The former appears to be a curved space analogue of the superHiggs effect. The F-term case is similar to the global case in the presence of supersymmetry breaking terms. There it was found that supersymmetry breaking resulted in the mixing of zero modes, which aided their destruction. In the case of supergravity with F-terms, the gravitino mixes the left and right moving fermions in the Lagrangian, resulting in the absence of zero modes.

Usually fermion zero modes are confined to the string by Yukawa couplings to the string Higgs fields. However in some cases the string gauge fields and gravity is enough to confine the fermions (this is only relevant for models with fermions which are massless outside the string). The wavefunctions of such zero modes have power law decay outside the string (as opposed to the usual exponential decay). Because of this the corresponding currents are less effective at stabilising string loops, and so the resulting vorton constraints are weaker than usual.

Finally one would like to include a gravitino mass for D-term string models and take into account the effects of supersymmetry breaking. As this is only possible for quantised Fayet-Iliopoulos terms $\xi = p/n$ (with p integer), and since the deficit angle at infinity implies that $0 < C_1 = 1 - p < 1$, it seems that D-term strings are difficult to construct when a gravitino mass is present.

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